

# ON ORDERS OF APPROXIMATION FUNCTIONS OF GENERALIZED SMOOTHNES IN LORENTZ SPACES

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**Abstract.** This paper considers the Lorentz space with mixed norm of periodic functions of many variables and of the generalized Nikol'skii – Besov classes. Estimates for the order of approximation of the generalized Nikol'skii – Besov classes by partial sums of Fourier's series for multiple trigonometric system in Lorentz spaces with mixed norm are obtained.

**Keywords:** Lorentz space, Nikol'skii – Besov class, approximations of functions, hyperbolic cross.

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## 1. INTRODUCTION

Let  $\bar{x} = (x_1, \dots, x_m) \in \mathbb{I}^m = [0, 2\pi]^m$  and let  $\theta_j, p_j \in [1, +\infty)$ ,  $j = 1, \dots, m$ ,  $\mathbb{N}$  be the set of natural numbers.

We shall denote by  $L_{\bar{p}, \bar{\theta}}(\mathbb{I}^m)$  the Lorentz spaces with mixed norm of Lebesgue measurable functions  $f(\bar{x})$  defined on  $\mathbb{R}^m$  with of period  $2\pi$  for each variable such that

$$\|f\|_{\bar{p}, \bar{\theta}} = \|\dots\|f\|_{p_1, \theta_1} \dots \|_{p_m, \theta_m} < +\infty,$$

where

$$\|g\|_{p, \theta} = \left\{ \int_0^{2\pi} (g^*(t))^{\theta} t^{\frac{\theta}{p}-1} dt \right\}^{\frac{1}{\theta}},$$

where  $g^*$  is a non-increasing rearrangement of the function  $|g|$  (see [12]).

As we know, that in case when  $p_j = \theta_j$ ,  $j = 1, \dots, m$ , the space  $L_{\bar{p}, \bar{\theta}}(\mathbb{I}^m)$  coincides with the Lebesgue space  $L_{\bar{p}}(I^m)$  with mixed norm (for the definition see [21], p. 128):

$$\|f\|_{\bar{p}} = \left[ \int_0^{2\pi} \left[ \dots \left[ \int_0^{2\pi} |f(\bar{x})|^{p_1} dx_1 \right]^{\frac{p_2}{p_1}} \dots \right]^{\frac{p_m}{p_{m-1}}} dx_m \right]^{\frac{1}{p_m}}.$$

Let  $L_{\bar{q}, \bar{\theta}}^{\circ}(\mathbb{I}^m)$  be the set of functions  $f \in L_{\bar{q}, \bar{\theta}}(\mathbb{I}^m)$  such that

$$\int_0^{2\pi} f(\bar{x}) dx_j = 0, \quad \forall j = 1, \dots, m,$$

and let  $a_{\bar{n}}(f)$  be the Fourier coefficients of the function  $f \in L_1(\mathbb{I}^m)$  with respect to the multiple trigonometric system  $\{e^{i\langle \bar{n}, \bar{x} \rangle}\}_{\bar{n} \in \mathbb{Z}^m}$ . Then, we set

$$\delta_{\bar{s}}(f, \bar{x}) = \sum_{\bar{n} \in \rho(\bar{s})} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle},$$

where  $\langle \bar{y}, \bar{x} \rangle = \sum_{j=1}^m y_j x_j$ ,  $\rho(\bar{s}) = \{\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : 2^{s_j-1} \leq |k_j| < 2^{s_j}, j = 1, \dots, m\}$ .

A function  $\Omega(\bar{t}) = \Omega(t_1, \dots, t_m)$  is a function of mixed module continuity type of an order  $l \in \mathbb{N}$  if it satisfies the following conditions:

- 1)  $\Omega(\bar{t}) > 0$ ,  $t_j > 0$ ,  $j = 1, \dots, m$ ,  $\Omega(\bar{t}) = 0$ , if  $\prod_{j=1}^m t_j = 0$ ;
- 2)  $\Omega(\bar{t})$  increases in each variable;
- 3)  $\Omega(k_1 t_1, \dots, k_m t_m) \leq \left( \prod_{j=1}^m k_j \right)^l \Omega(t_1, \dots, t_m)$ ,  $k_j \in \mathbb{N}$ ,  $j = 1, \dots, m$ ;
- 4)  $\Omega(\bar{t})$  is continuous for  $t_j > 0$ ,  $j = 1, \dots, m$ .

Let us consider the following sets

$$\Gamma(\Omega, N) = \left\{ \bar{s} = (s_1, \dots, s_m) \in \mathbb{Z}_+^m : \Omega(2^{-s_1}, \dots, 2^{-s_m}) \geq \frac{1}{N} \right\},$$

$$Q(\Omega, N) = \bigcup_{\bar{s} \in \Gamma(\Omega, N)} \rho(\bar{s}),$$

$$\Gamma^\perp(\Omega, N) = \mathbb{Z}_+^m \setminus \Gamma(\Omega, N), \quad (1)$$

$$\Lambda(\Omega, N) = \Gamma^\perp(\Omega, N) \setminus \Gamma^\perp(\Omega, 2^l N). \quad (2)$$

It follows from (1) and (2) that  $\Lambda(\Omega, N) \subset \Gamma^\perp(\Omega, N)$  and

$$\frac{1}{2^l N} \leq \Omega(2^{-\bar{s}}) < \frac{1}{N} \quad (3)$$

for  $\bar{s} \in \Lambda(\Omega, N)$ . In [23], N.N. Pustovoitov proved that  $\Lambda(\Omega, N) \neq \emptyset$  and

$$|\Lambda(\Omega, N)| \asymp (\log_2 N)^{m-1}, \quad (4)$$

where  $|F|$  is the number of elements of the set  $F$ .

We will use the notation  $S_{Q(\Omega, N)}(f, \bar{x}) = \sum_{\bar{k} \in Q(\Omega, N)} a_{\bar{k}}(f) \cdot e^{i(\bar{k}, \bar{x})}$  for a partial sum of the Fourier series of a function  $f$ .

For a sequence of numbers we write  $\{a_{\bar{n}}\}_{\bar{n} \in \mathbb{Z}^m} \in l_{\bar{p}}$  if

$$\|\{a_{\bar{n}}\}_{\bar{n} \in \mathbb{Z}^m}\|_{l_{\bar{p}}} = \left\{ \sum_{n_m=-\infty}^{\infty} \left[ \dots \left[ \sum_{n_1=-\infty}^{\infty} |a_{\bar{n}}|^{p_1} \right]^{\frac{p_2}{p_1}} \dots \right]^{\frac{p_m}{p_{m-1}}} \right]^{\frac{1}{p_m}} < +\infty,$$

where  $\bar{p} = (p_1, \dots, p_m)$ ,  $1 \leq p_j < +\infty$ ,  $j = 1, 2, \dots, m$ .

For a given function of mixed module smoothness type  $\Omega(\bar{t})$  consider the generalized Nikol'skii – Besov class

$$S_{\bar{p}, \bar{\theta}, \bar{\tau}}^\Omega B = \left\{ f \in L_{\bar{p}, \bar{\theta}}^\circ(I^m) : \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \leq 1 \right\},$$

where  $\bar{p} = (p_1, \dots, p_m)$ ,  $\bar{\theta} = (\theta_1, \dots, \theta_m)$ ,  $\bar{\tau} = (\tau_1, \dots, \tau_m)$ ,  $1 < p_j < +\infty$ ,  $1 < \theta_j < \infty$ ,  $1 \leq \tau_j \leq +\infty$ ,  $j = 1, \dots, m$ , and  $\Omega(2^{-\bar{s}}) = \Omega(2^{-s_1}, \dots, 2^{-s_m})$ .

If  $\Omega(\bar{t}) = \prod_{j=1}^m t_j^{r_j}$ ,  $r_j > 0$ ,  $j = 1, \dots, m$ , then this class is denoted by  $S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{\tau}} B$ .

In case  $p_j = \theta_j = p$  and  $\Omega(\bar{t}) = \prod_{j=1}^m t_j^{r_j}$ ,  $r_j < l$ ,  $\tau_j = +\infty$ ,  $j = 1, \dots, m$ ,  $S_{\bar{p}, \bar{\theta}, \bar{\tau}}^\Omega B$  was defined by S.M. Nikol'skii [19], and for  $1 \leq \tau_j < +\infty$ ,  $j = 1, \dots, m$ , by T.I. Amanov [6] and P.I. Lizorkin, S.M. Nikol'skii [18].

As pointed out in [35], [?] one of the difficulties in the theory of approximation of functions of several variables is the choice of harmonics of the approximating polynomials. The first author, who suggested to approximate functions of several variables by polynomials with harmonics in hyperbolic crosses, was K.I. Babenko [7]. After that approximations of various classes of smooth functions by this method were considered by S.A. Telyakovskii [32], B.S. Mityagin [19], Ya. S. Bugrov [13], N.S. Nikol'skaya [22], E.M. Galeev [16, 17], V.N. Temlyakov [33, 34], Dinh Dung [15], A.R. DeVore, S.V. Konyagin

and V.N. Temlyakov [14], H. - J. Schmeisser and W. Sickel [29], W. Sickel and T. Ullrich [27], A.S. Romanyuk [25, 26].

For the generalized Besov class this problem was considered by N.N. Pustovoitov [23], [24], Sun Yongsheng and Wang Heping [31], D.B. Bazakhanov [9], M. Sikhov [28], and S.A. Stasyuk [30].

Exact orders of the approximation of the Nikol'skii–Besov classes in the metric of the Lorentz space were found by the author [1, 2] and K.A. Bekmaganbetov [10], [11].

An order of approximation of the class  $S_{\vec{p}, \vec{\theta}, \vec{\tau}}^{\vec{\tau}} B$  by partial Fourier sums  $S_n^{\vec{\gamma}}(f, \vec{x}) = \sum_{\langle \vec{s}, \vec{\gamma} \rangle < n} \delta_{\vec{s}}(f, \vec{x})$  was found in [1]. In [?] for class  $S_{\vec{p}, \vec{\theta}, \vec{\tau}}^{\Omega} B$  proved following statement.

**Theorem** (see [4]). Let  $1 \leq \theta_j^{(1)}, \theta_j^{(2)}, \tau_j < +\infty$ ,  $1 < p_j < q_j < \infty$ ,  $j = 1, \dots, m$ , and  $\Omega(\vec{t})$  be a function of mixed module continuity type of an order  $l$ , which satisfies the conditions  $(S)$  and  $(S_l)$ ,  $\alpha_j > \frac{1}{p_j} - \frac{1}{q_j}$ ,  $j = 1, \dots, m$ .

1) If  $1 \leq \theta_j^{(2)} < \tau_j < +\infty$ ,  $j = 1, \dots, m$ , then

$$\begin{aligned} \frac{1}{N} (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ \prod_{j=1}^m 2^{s_j \left( \frac{1}{p_j} - \frac{1}{q_j} \right)} \right\}_{\vec{s} \in \Lambda(N)} \right\|_{l_{\vec{\theta}(2)}} &<< \sup_{f \in S_{\vec{p}, \vec{\theta}(1), \vec{\tau}}^{\Omega} B} \|f - S_{Q(N)}(f)\|_{\vec{q}, \vec{\theta}^{(2)}} << \\ &<< \frac{1}{N} \left\| \left\{ \prod_{j=1}^m 2^{s_j \left( \frac{1}{p_j} - \frac{1}{q_j} \right)} \right\}_{\vec{s} \in \Lambda(N)} \right\|_{l_{\vec{\epsilon}}}, \end{aligned}$$

where  $\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_m)$ ,  $\epsilon_j = \frac{\tau_j \theta_j^{(2)}}{\tau_j - \theta_j^{(2)}}$ ,  $j = 1, \dots, m$ .

2) If  $\tau_j \leq \theta_j^{(2)}$ ,  $j = 1, \dots, m$ , then

$$\begin{aligned} \sup_{\vec{s} \in \Lambda(N)} \Omega(2^{-\vec{s}}) \prod_{j=1}^m 2^{s_j \left( \frac{1}{p_j} - \frac{1}{q_j} \right)} &<< \sup_{f \in S_{\vec{p}, \vec{\theta}(1), \vec{\tau}}^{\Omega} B} \|f - S_{Q(N)}(f)\|_{\vec{q}, \vec{\theta}^{(2)}} << \\ &<< \sup_{\vec{s} \in \Gamma^{\perp}(N)} \Omega(2^{-\vec{s}}) \prod_{j=1}^m 2^{s_j \left( \frac{1}{p_j} - \frac{1}{q_j} \right)}. \end{aligned}$$

The notation  $A(y) \asymp B(y)$  means that there exist positive constants  $C_1, C_2$  such that  $C_1 A(y) \leq B(y) \leq C_2 A(y)$ . If  $B \leq C_2 A$  or  $A \geq C_1 B$ , then we write  $B << A$  or  $A >> B$ .

The main aim of the present paper is to estimate the order of the quantity

$$\sup_{f \in S_{\vec{p}, \vec{\theta}, \vec{\tau}}^{\Omega} B} \|f - S_{Q(\Omega, N)}(f)\|_{\vec{q}, \vec{\theta}}.$$

This paper is organized as follows. In the second section some auxiliary lemmas are given. The third section establishes the estimate of the order approximation of the Nikol'skii–Besov classes in the Lorentz space with mixed norm.

## 2. AUXILIARY LEMMAS

In what follows, we denote by  $\chi_{\varkappa(n)}(\vec{s})$  the characteristic function of the set  $\varkappa(n) = \{\vec{s} = (s_1, \dots, s_m) \in \mathbb{Z}_+^m : \langle \vec{s}, \vec{\gamma} \rangle = n\}$ .

**Lemma 1.** Let  $\bar{\tau} = (\tau_1, \dots, \tau_m)$ ,  $1 \leq \tau_j < +\infty$ ,  $j = 1, \dots, m$ . Then the following relation holds:

$$\left\| \left\{ \chi_{\kappa(n)}(\bar{s}) \right\}_{\bar{s} \in \kappa(n)} \right\|_{l_{\bar{\tau}}} \asymp n^{\sum_{j=2}^m \frac{1}{\tau_j}}.$$

**Lemma 2.** Let  $\bar{\gamma} = (\gamma_1, \dots, \gamma_m)$ ,  $\bar{\gamma}' = (\gamma'_1, \dots, \gamma'_m)$ ,  $\gamma'_j = \gamma_j$ ,  $j = 1, \dots, \nu$ ,  $1 < \gamma_j < \gamma'_j$ ,  $j = \nu + 1, \dots, m$ , and let  $\bar{\tau} = (\tau_{l+1}, \dots, \tau_m)$ , where  $1 \leq \tau_j < +\infty$ ,  $j = 1, \dots, m$ , and  $\alpha > 0$ . Then the following relation holds:

$$I_n^{(l)} = \left\| \left\{ 2^{-\alpha \langle \bar{s}, \bar{\gamma}' \rangle} \right\}_{\bar{s} \in \kappa(n)} \right\|_{l_{\bar{\tau}}} \asymp 2^{-n\alpha} \cdot n^{\sum_{j=2}^{\nu} \frac{1}{\tau_j}}.$$

Lemma 1, 2 are proved in [2].

Let us recall definitions of the conditions  $(S)$ ,  $(S_l)$  given by S.B.Stechkin and N.K. Bary [8].

**Definition.** A function  $g(t)$  satisfies the condition  $(S)$ , if for some  $\alpha \in (0, 1)$  the function  $t^{-\alpha}g(t)$  almost increases on  $(0, 1]$ .

We say that a function  $\Omega(\bar{t})$  satisfies the condition  $(S)$  on  $(0, 1]^m$ , if it satisfies this condition on each variable.

**Definition.** A function  $g(t)$  satisfies the condition  $(S_l)$ , if for some  $\alpha \in (0, l)$  the function  $t^{-\alpha}g(t)$  almost decreases on  $(0, 1]$ .

We say that a function  $\Omega(\bar{t})$  satisfies the condition  $(S_l)$  on  $(0, 1]^m$ , if it satisfies this condition on each variable.

**Lemma 3.** (see [4]) Let  $1 \leq \theta_j < +\infty$ ,  $j = 1, \dots, m$ , and  $\Omega(\bar{t})$  be a function of mixed module continuity type of an order  $l$  which satisfies the  $(S)$ -condition for  $\bar{\alpha} = (\alpha_1, \dots, \alpha_m)$ ,  $\alpha_j > \beta_j \geq 0$ ,  $j = 1, \dots, m$ . Then for  $1 \leq \theta_j < +\infty$ ,  $j = 1, \dots, m$ , the following relation holds

$$\begin{aligned} & \left\| \left\{ \Omega(2^{-s_1}, \dots, 2^{-s_m}) \prod_{j=1}^m 2^{s_j \beta_j} \right\}_{\bar{s} \in \Gamma^\perp(\Omega, N)} \right\|_{l_{\bar{\theta}}} \asymp \\ & \asymp \left\| \left\{ \Omega(2^{-s_1}, \dots, 2^{-s_m}) \prod_{j=1}^m 2^{s_j \beta_j} \right\}_{\bar{s} \in \Lambda(N)} \right\|_{l_{\bar{\theta}}}. \end{aligned}$$

**Lemma 4.** (see [4]). Let  $\Omega(\bar{t})$  be a function of mixed module continuity type of an order  $l$ , which satisfies the conditions  $(S)$  and  $(S_l)$ ,  $1 \leq \tau_j < +\infty$ ,  $j = 1, \dots, m$ , and  $\Lambda(\Omega, N) = \Gamma^\perp(\Omega, N) \setminus \Gamma^\perp(\Omega, 2^l N)$ . Then

$$\left\| \left\{ \chi_{\Lambda(\Omega, N)}(\bar{s}) \right\}_{\bar{s} \in \Lambda(\Omega, N)} \right\|_{l_{\bar{\tau}}} \asymp (\log_2 N)^{\sum_{j=2}^m \frac{1}{\tau_j}}.$$

**Remark.** Note that for the case  $\tau_1 = \dots = \tau_m = 1$  Lemma 4 was proved by N.N. Pustovoytov [23].

**Theorem 1.** Let  $\bar{q} = (q_1, \dots, q_m)$ ,  $1 < q_j < \infty$ ,  $j = 1, \dots, m$ ,  $\beta = \min\{q_1, \dots, q_m, 2\}$ . Then, for any function  $f \in L_{\bar{q}}(I^m)$ , the following inequality holds

$$\|f\|_{\bar{q}} < \left\{ \sum_{\bar{s} \in \mathbb{Z}_+^m} \|\delta_{\bar{s}}(f)\|_{\bar{q}}^\beta \right\}^{\frac{1}{\beta}}.$$

The proof of theorem is given in [3].

**Theorem 2.** (see [1]). Let  $\bar{p} = (p_1, \dots, p_m)$ ,  $\bar{q} = (q_1, \dots, q_m)$ ,  $\bar{\theta}^{(1)} = (\theta_1^{(1)}, \dots, \theta_m^{(1)})$ ,  $\bar{\theta}^{(2)} = (\theta_1^{(2)}, \dots, \theta_m^{(2)})$ . Assume that  $1 \leq p_j < q_j < +\infty$ ,  $1 \leq \theta_j^{(1)}, \theta_j^{(2)} < +\infty$ ,  $j = 1, \dots, m$ . If  $f \in \mathring{L}_{\bar{p}, \bar{\theta}^{(1)}}(\mathbb{I}^m)$ ,  $\max_{j=1, \dots, m-1} \theta_j^{(2)} < \min_{j=2, \dots, m} q_j$  and the quantity

$$\sigma(f) \equiv \left\{ \sum_{s_m=1}^{\infty} 2^{s_m \theta_m^{(2)} (\frac{1}{p_m} - \frac{1}{q_m})} \left[ \dots \left[ \sum_{s_1=1}^{\infty} 2^{s_1 \theta_1^{(2)} (\frac{1}{p_1} - \frac{1}{q_1})} \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}}^{\theta_1^{(2)}} \right]^{\frac{\theta_2^{(2)}}{\theta_1^{(2)}}} \dots \right]^{\frac{\theta_m^{(2)}}{\theta_{m-1}^{(2)}}} \right\}^{\frac{1}{\theta_m^{(2)}}}$$

is finite, then  $f \in L_{\bar{q}, \bar{\theta}^{(2)}}^{\circ}(\mathbb{I}^m)$  and

$$\|f\|_{\bar{q}, \bar{\theta}^{(2)}} << \sigma(f).$$

**Theorem 3.** (see [1]). Let  $\bar{q} = (q_1, \dots, q_m)$ ,  $\bar{\theta} = (\theta_1, \dots, \theta_m)$ ,  $\bar{\lambda} = (\lambda_1, \dots, \lambda_m)$ . Assume that  $1 < q_j < \tau_j < +\infty$ ,  $1 < \theta_j < +\infty$ ,  $j = 1, \dots, m$ . If  $f \in \mathring{L}_{\bar{q}, \bar{\theta}}(\mathbb{I}^m)$  and

$$f(\bar{x}) \sim \sum_{\bar{s} \in \mathbb{Z}_+^m} b_{\bar{s}} \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle},$$

then

$$\|f\|_{\bar{q}, \bar{\theta}} >> \left\{ \sum_{s_m=1}^{\infty} 2^{s_m \theta_m (\frac{1}{\lambda_m} - \frac{1}{q_m})} \left[ \dots \left[ \sum_{s_1=1}^{\infty} 2^{s_1 \theta_1 (\frac{1}{\lambda_1} - \frac{1}{q_1})} (\|\delta_{\bar{s}}(f)\|_{\bar{\lambda}, \bar{\theta}})^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \dots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}.$$

### 3. MAIN RESULTS

Let us prove the main results of the present paper.

Consider the function  $\Omega_1(\bar{t}) = \Omega(\bar{t}) \prod_{j=1}^m t_j^{-\frac{1}{p_j} - \frac{1}{q_j}}$ ,  $t_j \in (0, 1]$ ,  $j = 1, \dots, m$  and respectively

set  $Q(\Omega_1, N)$ ,  $\Gamma^{\perp}(\Omega_1, N)$ ,  $\Lambda(\Omega_1, N)$ .

**Theorem 4.** Let  $1 \leq \theta_j^{(1)}, \theta_j^{(2)}, \tau_j < +\infty$ ,  $1 < p_j < q_j < \infty$ ,  $j = 1, \dots, m$ , and  $\Omega(\bar{t})$  be a function of mixed module continuity type of an order  $l$ , which satisfies the conditions (S)

and  $(S_l)$ ,  $\alpha_j > \frac{1}{p_j} - \frac{1}{q_j}$ ,  $j = 1, \dots, m$   $\Omega_1(\bar{t}) = \Omega(\bar{t}) \prod_{j=1}^m t_j^{-\frac{1}{p_j} - \frac{1}{q_j}}$ .

1) If  $1 \leq \theta_j^{(2)} < \tau_j < +\infty$ ,  $j = 1, \dots, m$ , then

$$\sup_{f \in S_{\bar{p}, \bar{\theta}^{(1)}}^{\Omega} B} \|f - S_{Q(\Omega_1, N)}(f)\|_{\bar{q}, \bar{\theta}^{(2)}} \asymp \frac{1}{N} (\log_2 N)^{\sum_{j=2}^m (\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j})}.$$

2) If  $\tau_j \leq \theta_j^{(2)}$ ,  $j = 1, \dots, m$ , then

$$\sup_{f \in S_{\bar{p}, \bar{\theta}^{(1)}}^{\Omega} B} \|f - S_{Q(\Omega_1, N)}(f)\|_{\bar{q}, \bar{\theta}^{(2)}} \asymp \frac{1}{N}.$$

**Proof.** Taking into account  $\delta_{\bar{s}}(f - S_{Q(\Omega_1, N)}(f)) = 0$ , if  $\bar{s} \in Q(\Omega_1, N)$  and  $\delta_{\bar{s}}(f - S_{Q(\Omega_1, N)}(f)) = \delta_{\bar{s}}(f)$ , if  $\bar{s} \notin Q(\Omega_1, N)$  By Theorem 2, we have

$$\begin{aligned} \|f - S_{Q(\Omega_1, N)}(f)\|_{\bar{q}, \bar{\theta}^{(2)}} &<< \left\| \left\{ \prod_{j=1}^m 2^{s_j \left( \frac{1}{p_j} - \frac{1}{q_j} \right)} \|\delta_{\bar{s}}(f - S_{Q(\Omega_1, N)}(f))\|_{\bar{p}, \bar{\theta}^{(1)}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}^{(2)}}} = \\ &= C \left\| \left\{ \prod_{j=1}^m 2^{s_j \left( \frac{1}{p_j} - \frac{1}{q_j} \right)} \|\delta_{\bar{s}}(f - S_{Q(\Omega_1, N)}(f))\|_{\bar{p}, \bar{\theta}^{(1)}} \right\}_{\bar{s} \in \Gamma^\perp(\Omega_1, N)} \right\|_{l_{\bar{\theta}^{(2)}}}. \end{aligned}$$

for any function  $f \in S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^\Omega B$ .

Since  $\beta_j = \frac{\tau_j}{\theta_j^{(2)}} > 1, j = 1, \dots, m$ , and by applying Holder's inequality we obtain the following

$$\begin{aligned} \|f - S_{Q(\Omega_1, N)}(f)\|_{\bar{q}, \bar{\theta}^{(2)}} &<< \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}^{(1)}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \times \\ &\times \left\| \left\{ \Omega(2^{-\bar{s}}) \prod_{j=1}^m 2^{s_j \left( \frac{1}{p_j} - \frac{1}{q_j} \right)} \right\}_{\bar{s} \in \Gamma^\perp(\Omega_1, N)} \right\|_{l_{\bar{\epsilon}}}, \end{aligned} \quad (5)$$

where  $\bar{\epsilon} = (\epsilon_1, \dots, \epsilon_m)$ ,  $\epsilon_j = \frac{\tau_j \theta_j^{(2)}}{\tau_j - \theta_j^{(2)}}, j = 1, \dots, m$ .

2. Since by assumption theorem the function  $\Omega(\bar{t})$  satisfies  $S$  and  $S_l$  conditions and  $\alpha_j > \frac{1}{p_j} - \frac{1}{q_j}, j = 1, \dots, m$ , then the function  $\Omega_1(\bar{t})$  satisfies conditions  $S$  and  $S_l$ .

Therefore, by Lemma 3, Lemma 4 and the definition of the set  $\Gamma^\perp(\Omega_1, N)$  in (5), we have

$$\begin{aligned} \sup_{f \in S_{\bar{p}, \bar{\theta}^{(1)}, \bar{\tau}}^\Omega B} \|f - S_{Q(\Omega_1, N)}(f)\|_{\bar{q}, \bar{\theta}^{(2)}} &<< \left\| \{\Omega_1(2^{\bar{s}})\}_{\bar{s} \in \Lambda(\Omega_1, N)} \right\|_{l_{\bar{\epsilon}}} << \\ &<< \frac{1}{N} \left\| \{\chi_{\Lambda(\Omega_1, N)}(\bar{s})\}_{\bar{s} \in \Lambda(\Omega_1, N)} \right\|_{l_{\bar{\epsilon}}} << \frac{1}{N} (\log_2 N)^{\sum_{j=2}^m \left( \frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j} \right)}. \end{aligned}$$

In item 1) of the theorem the upper bound has been proved.

Let us prove the lower bound. Consider the function

$$f_0(\bar{x}) = (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} \sum_{\bar{s} \in \Lambda(\Omega_1, N)} \prod_{j=1}^m \Omega(2^{-\bar{s}}) 2^{-s_j \left( 1 - \frac{1}{p_j} \right)} \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

In one-dimensional case for Dirichlet's kernel  $D_n(x) = \frac{1}{2} + \sum_{k=1}^n e^{ikx}$  the following statement holds

$$\|D_n\|_{p, \theta} \asymp n^{1 - \frac{1}{p}}, \quad 1 < p < +\infty, \quad 1 < \theta < +\infty.$$

Then, by the property of the norm, we have

$$\left\| \sum_{k_j=2^{s_j-1}}^{2^{s_j}-1} e^{ik_j x_j} \right\|_{p_j, \theta_j^{(1)}} \leq \|D_{2^{s_j}-1}\|_{p_j, \theta_j^{(1)}} + \|D_{2^{s_j-1}-1}\|_{p_j, \theta_j^{(1)}} << 2^{s_j \left( 1 - \frac{1}{p_j} \right)},$$

provided  $1 < p_j < +\infty$ ,  $1 < \theta_j^{(1)} < +\infty$ ,  $j = 1, \dots, m$ . Hence

$$\left\| \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \bar{\theta}^{(1)}} = \prod_{j=1}^m \left\| \sum_{k_j=2^{s_j}-1} e^{ik_j x_j} \right\|_{p_j, \theta_j^{(1)}} << \prod_{j=1}^m 2^{s_j \left(1 - \frac{1}{p_j}\right)}.$$

Let us prove the rest of the equality. By Lemma B in [1], the following inequality holds

$$\max_{\bar{x} \in I^m} \left| \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle} \right| << \prod_{j=1}^m 2^{s_j \left(1 - \frac{1}{p_j}\right)} \left\| \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \bar{\theta}^{(1)}}. \quad (6)$$

It is known that

$$\max_{\bar{x} \in I^m} \left| \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle} \right| \geq \left| \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{0} \rangle} \right| \geq 2^{-m} \prod_{j=1}^m 2^{s_j}.$$

Therefore, it follows from (6) that

$$\prod_{j=1}^m 2^{s_j \left(1 - \frac{1}{p_j}\right)} << \left\| \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \bar{\theta}^{(1)}}.$$

Thus, we have proved the relation

$$\left\| \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \bar{\theta}^{(1)}} \asymp \prod_{j=1}^m 2^{s_j \left(1 - \frac{1}{p_j}\right)}. \quad (7)$$

Therefore, by Lemma 4 and by estimation (7), we have

$$\begin{aligned} & \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f_0)\|_{\bar{p}, \bar{\theta}^{(1)}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} = \\ & = \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} \Omega(2^{-\bar{s}}) \prod_{j=1}^m 2^{-s_j \left(1 - \frac{1}{p_j}\right)} \left\| \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \bar{\theta}^{(1)}} \right\}_{\bar{s} \in \Lambda(\Omega_1, N)} \right\|_{l_{\bar{\tau}}} \\ & = (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ \chi_{\Lambda(\Omega_1, N)}(\bar{s}) \right\}_{\bar{s} \in \Lambda(\Omega_1, N)} \right\|_{l_{\bar{\tau}}} \leq C_0. \end{aligned}$$

Hence  $C_0^{-1} f_0 \in S_{\bar{p}, \bar{\tau}}^{\Omega} B$ . Now taking into account that  $S_{Q(\Omega_1, N)}^{\bar{\gamma}}(f_0, \bar{x}) = 0$ ,  $\bar{x} \in I^m$  and using Theorem 4 and (7), Lemma 4, we obtain

$$\begin{aligned} & \|f_0 - S_{Q(\Omega_1, N)}(f_0)\|_{\bar{q}, \bar{\theta}^{(2)}} = \|f_0\|_{\bar{q}, \bar{\theta}^{(2)}} >> \\ & >> \left\| \left\{ \prod_{j=1}^m 2^{s_j \left(\frac{1}{\lambda_j} - \frac{1}{q_j}\right)} \|\delta_{\bar{s}}(f_0)\|_{\bar{\lambda}, \bar{\theta}^{(1)}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\theta}^{(2)}}} = \\ & = C \left\| \left\{ \prod_{j=1}^m 2^{s_j \left(\frac{1}{\lambda_j} - \frac{1}{q_j}\right)} (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} \Omega(2^{-\bar{s}}) \prod_{j=1}^m 2^{-s_j \left(1 - \frac{1}{p_j}\right)} \left\| \sum_{\bar{k} \in \rho(\bar{s})} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{\lambda}, \bar{\theta}^{(1)}} \right\}_{\bar{s} \in \Lambda(\Omega_1, N)} \right\|_{l_{\bar{\theta}^{(2)}}} >> \\ & >> (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ \Omega_1(2^{-\bar{s}}) \right\}_{\bar{s} \in \Lambda(\Omega_1, N)} \right\|_{l_{\bar{\theta}^{(2)}}} >> \end{aligned}$$

$$>> \frac{1}{N} (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \left\{ \chi_{\Lambda(\Omega_1, N)} \right\}_{\bar{s} \in \Lambda(\Omega_1, N)} \right\|_{l_{\bar{\theta}(2)}} >> \frac{1}{N} (\log_2 N)^{\sum_{j=2}^m (\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j})}.$$

Thus,

$$\sup_{f \in S_{\bar{p}, \bar{\theta}(1), \bar{\tau}}^\Omega B} \|f - S_{Q(\Omega_1, N)}(f)\|_{\bar{q}, \bar{\theta}(2)} >> \frac{1}{N} (\log_2 N)^{\sum_{j=2}^m (\frac{1}{\theta_j^{(2)}} - \frac{1}{\tau_j})}.$$

Item 1) of the theorem has been proved.

Let us prove item 2) of the theorem. Since  $\tau_j \leq \theta_j^{(2)}$ ,  $j = 1, \dots, m$ , then by applying Theorem 2 and Jensen's inequality (see [21], p. 125), we obtain

$$\begin{aligned} \|f - S_{Q(\Omega_1, N)}(f)\|_{\bar{q}, \bar{\theta}(2)} &<< \left\| \left\{ \prod_{j=1}^m 2^{s_j \left( \frac{1}{p_j} - \frac{1}{q_j} \right)} \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}(1)} \right\}_{\bar{s} \in \Gamma^\perp(\Omega_1, N)} \right\|_{l_{\bar{\tau}}} << \\ &<< \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}(1)} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \sup_{\bar{s} \in \Gamma^\perp(\Omega_1, N)} \Omega(2^{-\bar{s}}) \prod_{j=1}^m 2^{s_j \left( \frac{1}{p_j} - \frac{1}{q_j} \right)} << \frac{1}{N} \end{aligned}$$

for any function  $f \in S_{\bar{p}, \bar{\theta}(1), \bar{\tau}}^\Omega B$ , which proves the upper bound in item 2). For the lower bound, consider the function

$$f_1(\bar{x}) = \Omega(2^{-\tilde{s}}) 2^{-\sum_{j=1}^m \tilde{s}_j \left( 1 - \frac{1}{p_j} \right)} \sum_{\bar{k} \in \rho(\tilde{s})} e^{i\langle \bar{k}, \bar{x} \rangle},$$

where  $\tilde{s} = (\tilde{s}_1, \dots, \tilde{s}_m) \in \Lambda(\Omega_1, N)$ . Then  $f_1 \in S_{\bar{p}, \bar{\theta}(1), \bar{\tau}}^\Omega B$ . Next, by (7), we have

$$\begin{aligned} \|f_1 - S_{Q(\Omega_1, N)}(f_1)\|_{\bar{q}, \bar{\theta}(2)} &= \|f_1\|_{\bar{q}, \bar{\theta}(2)} >> \\ &>> \Omega(2^{-\tilde{s}}) 2^{-\sum_{j=1}^m \tilde{s}_j \left( 1 - \frac{1}{p_j} \right)} \prod_{j=1}^m 2^{\tilde{s}_j \left( 1 - \frac{1}{q_j} \right)} = C \Omega(2^{-\tilde{s}}) \prod_{j=1}^m 2^{\tilde{s}_j \left( \frac{1}{p_j} - \frac{1}{q_j} \right)} \end{aligned}$$

for  $\tilde{s} \in \Lambda(\Omega_1, N)$ .

Hence, by (3) we obtain

$$\sup_{f \in S_{\bar{p}, \bar{\theta}(1), \bar{\tau}}^\Omega B} \|f - S_{Q(\Omega_1, N)}(f)\|_{\bar{q}, \bar{\theta}(2)} >> \frac{1}{N}.$$

This proves the lower bound in item 2).

**Theorem 5.** Let  $\Omega(\bar{t})$  be a function of mixed module continuity type of an order  $l$  which satisfies the conditions (S) and (S<sub>l</sub>),  $1 < q_j < p_j < \infty$ ,  $p_j \geq 2$ ,  $1 < \theta_j < \infty$ ,  $1 \leq \tau_j \leq +\infty$ ,  $j = 1, \dots, m$ .

1) If  $2 < \tau_j < +\infty$ ,  $j = 1, \dots, m$ , then

$$\sup_{f \in S_{\bar{p}, \bar{\tau}}^\Omega B} \|f - S_{Q(\Omega, N)}(f)\|_{\bar{q}, \bar{\theta}} \asymp \frac{1}{N} (\log_2 N)^{\sum_{j=2}^m \left( \frac{1}{2} - \frac{1}{\tau_j} \right)}.$$

2) If  $\tau_j \leq 2$ ,  $j = 1, \dots, m$ , then

$$\sup_{f \in S_{\bar{p}, \bar{\tau}}^\Omega B} \|f - S_{Q(\Omega, N)}(f)\|_{\bar{q}, \bar{\theta}} \asymp \frac{1}{N}.$$



3) If  $1 < q_j < p_j \leq 2$ ,  $j = 1, \dots, m$  and  $p_0 = \min\{p_1, \dots, p_m\} < \tau_j$ ,  $j = 1, \dots, m$ , then

$$\frac{1}{N}(\log_2 N)^{\sum_{j=2}^m \left(\frac{1}{p_j} - \frac{1}{\tau_j}\right)} << \sup_{f \in S_{\bar{p}, \bar{\tau}}^\Omega B} \|f - S_{Q(\Omega, N)}(f)\|_{\bar{q}, \bar{\theta}} << \frac{1}{N}(\log_2 N)^{\sum_{j=2}^m \left(\frac{1}{p_0} - \frac{1}{\tau_j}\right)}.$$

**Proof.** Since  $q_j < p_j$ ,  $j = 1, \dots, m$ , then  $L_{\bar{p}}(\mathbb{I}^m) \subset L_{\bar{q}, \bar{\theta}}(\mathbb{I}^m)$  and we have

$$\|f\|_{\bar{q}, \bar{\theta}} << \|f\|_{\bar{p}}, \quad f \in L_{\bar{p}}(\mathbb{I}^m).$$

Therefore  $S_{\bar{p}, \bar{\tau}}^\Omega B \subset L_{\bar{q}, \bar{\theta}}(\mathbb{I}^m)$  and

$$\begin{aligned} \|f - S_{Q(\Omega, N)}(f)\|_{\bar{q}, \bar{\theta}} &<< \|f - S_{Q(\Omega, N)}(f)\|_{\bar{p}} = \\ &= C \left\| \sum_{\bar{s} \in \Gamma^\perp(\Omega, N)} \delta_{\bar{s}}(f) \right\|_{\bar{p}}. \end{aligned} \quad (8)$$

for any function  $f \in S_{\bar{p}, \bar{\tau}}^\Omega B$ .

Now, since  $2 \leq p_j < +\infty$ ,  $j = 1, \dots, m$ , using Theorem 1 from (8) we obtain

$$\begin{aligned} \|f - S_{Q(\Omega, N)}(f)\|_{\bar{q}, \bar{\theta}} &<< \left\{ \sum_{\bar{s} \in \Gamma^\perp(\Omega, N)} \|\delta_{\bar{s}}(f)\|_{\bar{p}}^2 \right\}^{\frac{1}{2}} = \\ &= C \left\{ \sum_{\bar{s} \in \Gamma^\perp(\Omega, N)} \Omega^2(2^{-\bar{s}}) \left( \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right)^2 \right\}^{\frac{1}{2}} \end{aligned} \quad (9)$$

for any function  $f \in S_{\bar{p}, \bar{\tau}}^\Omega B$ .

Item 1) proved in [4].

Let us prove item 2). If  $\tau_j \leq 2$ ,  $j = 1, \dots, m$ , then using Jensen's inequality we have

$$\begin{aligned} &\left\{ \sum_{\bar{s} \in \Gamma^\perp(\Omega, N)} \|\delta_{\bar{s}}(f)\|_{\bar{p}}^2 \right\}^{\frac{1}{2}} << \\ &<< \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \sup_{\bar{s} \in \Gamma^\perp(\Omega, N)} \Omega(2^{-\bar{s}}). \end{aligned}$$

Therefore, from the inequality (9) we obtain

$$\sup_{f \in S_{\bar{p}, \bar{\tau}}^\Omega B} \|f - S_{Q(\Omega, N)}(f)\|_{\bar{q}, \bar{\theta}} << \sup_{\bar{s} \in \Gamma^\perp(\Omega, N)} \Omega(2^{-\bar{s}}) << \frac{1}{N},$$

in case  $2 < p_j < +\infty$ ,  $\tau_j \leq 2$ ,  $j = 1, \dots, m$ . This proves the upper bound. The lower bound in item 2) proved in [4].

Let us prove item 3).

Since  $1 < p_j \leq 2$ ,  $j = 1, \dots, m$ , using Theorem 1 from (8) we obtain

$$\begin{aligned} \|f - S_{Q(\Omega, N)}(f)\|_{\bar{q}, \bar{\theta}} &<< \left\{ \sum_{\bar{s} \in \Gamma^\perp(\Omega, N)} \|\delta_{\bar{s}}(f)\|_{\bar{p}}^{p_0} \right\}^{\frac{1}{p_0}} = \\ &= C \left\{ \sum_{\bar{s} \in \Gamma^\perp(\Omega, N)} \Omega^{p_0}(2^{-\bar{s}}) \left( \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right)^{p_0} \right\}^{\frac{1}{p_0}} \end{aligned} \quad (10)$$

for any function  $f \in S_{\bar{p}, \bar{\tau}}^\Omega B$ .

If  $p_0 < \tau_j < +\infty$ ,  $j = 1, \dots, m$ , then by Holder's inequality from (10), we get

$$\begin{aligned} \|f - S_{Q(\Omega, N)}(f)\|_{\bar{q}, \bar{\theta}} &<< \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \times \\ &\times \left\| \left\{ \Omega(2^{-\bar{s}}) \right\}_{\bar{s} \in \Gamma^\perp(\Omega, N)} \right\|_{l_{\bar{\epsilon}}}, \end{aligned} \quad (11)$$

where  $\bar{\epsilon} = (\epsilon_1, \dots, \epsilon_m)$ ,  $\epsilon_j = 2\beta'_j$ ,  $\frac{1}{\beta_j} + \frac{1}{\beta'_j} = 1$ ,  $\beta_j = \frac{\tau_j}{p_0}$ ,  $j = 1, \dots, m$ .

Now by Lemma 3 and 4 from (11) we obtain

$$\|f - S_{Q(\Omega, N)}(f)\|_{\bar{q}, \bar{\theta}} << \frac{1}{N} (\log_2 N)^{\sum_{j=2}^m \left( \frac{1}{p_0} - \frac{1}{\tau_j} \right)}$$

for any function  $f \in S_{\bar{p}, \bar{\tau}}^\Omega B$ . This proves the upper bound.

Let us prove the lower bound. consider the set similarly in [23]

$$\Lambda'(\Omega, N) = \left\{ \bar{s} \in \Lambda(\Omega, N) : s_j > \frac{1}{2ml} \log_2(C_3 N), j = 1, \dots, m \right\}.$$

N.N. Pustovoitov [23] has been proved that, number of point is equal to  $|\Lambda'(\Omega, N)| \asymp (\log_2 N)^{m-1}$ .

After this we choose set  $\Lambda(\bar{\Omega}, N)$ . Lets take a number  $v = [|\Lambda'(\Omega, N)|^{\frac{1}{m}}]$  - which is whole part of a number  $|\Lambda'(\Omega, N)|^{\frac{1}{m}}$ . Divide set  $\mathbb{I}^m = [-\pi, \pi]^m$  to  $v^m$  cubes with side equal to  $\frac{2\pi}{v}$ . Then choose set  $\Lambda(\bar{\Omega}, N) \subset \Lambda(\Omega, N)$ , such that  $|\Lambda(\bar{\Omega}, N)| = v^m$ , and define bijection between this set  $\Lambda(\bar{\Omega}, N)$  and the set of cubes.

Let for  $\bar{s} \in \Lambda(\bar{\Omega}, N)$  point  $\bar{x}^{\bar{s}}$  denote centre of the cube. Further we set notation

$$u = \left[ 2^{\frac{1}{m-1} \sum_{j=2}^m (1 - \frac{1}{p_j}) (\sum_{j=1}^m (1 - \frac{1}{p_j}))^{-1} \log_2 |\Lambda(\Omega, N)|} \right].$$

Consider the function

$$f_3(\bar{x}) = \frac{1}{N} (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j} - \sum_{j=1}^m (1 - \frac{1}{p_j})} \Psi(\bar{x}),$$

where (see [23])

$$\Psi(\bar{x}) = \sum_{\bar{s} \in \Lambda(\bar{\Omega}, N)} e^{i\langle \bar{k}^{\bar{s}}, \bar{x} - \bar{x}^{\bar{s}} \rangle} K_u(\bar{x} - \bar{x}^{\bar{s}}), \quad \bar{k}^{\bar{s}} = (k_1^{\bar{s}}, \dots, k_m^{\bar{s}}), k_j^{\bar{s}} = 2^{s_j} + 2^{s_j-1}, j = 1, \dots, m,$$

$$K_u(\bar{x}) = 2^m \prod_{j=1}^m K_u(x_j),$$

$K_u(x_j)$  - is Fejer core of order  $u$  by variable  $x_j$ ,  $j = 1, \dots, m$ . Note that,

$$u \asymp (\log_2 N)^{\sum_{j=2}^m (1 - \frac{1}{p_j}) (\sum_{j=1}^m (1 - \frac{1}{p_j}))^{-1}}. \quad (13)$$

In [23] has been proved that

$$E_{Q(\Omega, N)}(\Psi)_1 \gg |\Lambda(\Omega, N)|. \quad (12)$$

Lets show that  $C_3 f_3 \in S_{\bar{p}, \bar{\tau}}^\Omega B$  for some constant  $C_3 > 0$ .

Since for Fejer core with one variable we have got estimation  $\|K_u(y)\|_p \asymp u^{1-\frac{1}{p}}$ ,  $1 \leq p \leq \infty$ ,

$$\|K_u(\bar{x})\|_{\bar{p}} \asymp u^{\sum_{j=1}^m (1 - \frac{1}{p_j})}.$$

Using this relation and  $|\Lambda'(\Omega, N)| \asymp |\Lambda(\Omega, N)| \asymp (\log_2 N)^{m-1}$  we get

$$\begin{aligned} & \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f_3)\|_{\bar{p}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} << \\ & << \frac{1}{N} (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j} u - \sum_{j=1}^m (1-\frac{1}{p_j})} \left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) u^{\sum_{j=1}^m (1-\frac{1}{p_j})} \right\}_{\bar{s} \in \Lambda(\bar{\Omega}, N)} \right\|_{l_{\bar{\tau}}} << \\ & << (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \{1\}_{\bar{s} \in \Lambda(\bar{\Omega}, N)} \right\|_{l_{\bar{\tau}}}. \end{aligned}$$

Since by Lemma 4 estimation

$$\left\| \{1\}_{\bar{s} \in \Lambda(\bar{\Omega}, N)} \right\|_{l_{\bar{\tau}}} << (\log_2 N)^{\sum_{j=2}^m \frac{1}{\tau_j}}$$

is true, then

$$\left\| \left\{ \Omega^{-1}(2^{-\bar{s}}) \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} << C.$$

Because  $C_3 f_3 \in S_{\bar{p}, \bar{\tau}}^{\Omega} B$ .

Since  $L_{\bar{q}, \bar{\theta}}(\mathbb{I}^m) \subset L_1(\mathbb{I}^m)$  and  $\|f\| << \|f\|_{\bar{q}, \bar{\theta}}$ , then

$$E_{Q(\Omega, N)}(f_3)_{\bar{q}, \bar{\theta}} >> E_{Q(\Omega, N)}(f_3)_1 = C \frac{1}{N} (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j} u - \sum_{j=1}^m (1-\frac{1}{p_j})} E_{Q(\Omega, N)}(\Psi)_1.$$

Therefore, by the estimates (12), (13) we have got

$$\begin{aligned} E_{Q(\Omega, N)}(f_3)_{\bar{q}, \bar{\theta}} &>> \frac{1}{N} (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j} u - \sum_{j=1}^m (1-\frac{1}{p_j})} |\Lambda(\Omega, N)| >> \\ &>> \frac{1}{N} (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} (\log_2 N)^{-\sum_{j=2}^m (1-\frac{1}{p_j})} (\log_2 N)^{m-1} = C \frac{1}{N} (\log_2 N)^{\sum_{j=2}^m (\frac{1}{p_j} - \frac{1}{\tau_j})}. \end{aligned}$$

The Theorem 5 is proved.

Now consider the case  $q_j = p_j$ ,  $j = 1, \dots, m$  and  $\Omega(\bar{t}) = \prod_{j=1}^m t_j^{r_j}$ ,  $r_j > 0$ ,  $t_j \in [0, 1]$ ,  $j = 1, \dots, m$ .

**Theorem 6.** Let  $\bar{r} = (r_1, \dots, r_m)$ ,  $0 < r_1 = \dots = r_\nu < r_{\nu+1} \leq \dots \leq r_m$  and  $1 < q_j < p_j < \infty$ ,  $p_j \geq 2$ ,  $1 < \theta_j < \infty$ ,  $1 \leq \tau_j \leq +\infty$ ,  $j = 1, \dots, m$ .

If  $2 \leq p_j < \theta_j < \infty$ ,  $2 \leq \tau_j < +\infty$ ,  $j = 1, \dots, m$ , then

$$\sup_{f \in S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B} \|f - S_N^\gamma(f)\|_{\bar{p}} << N^{-r_1} (\log_2 N)^{\sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{\theta_j}\right)} (\log_2 N)^{\sum_{j=2}^m \left(\frac{1}{2} - \frac{1}{\tau_j}\right)}$$

and if  $p_1 = \dots = p_m = p$ , then

$$N^{-r_1} (\log_2 N)^{\sum_{j=1}^m \left(\frac{1}{p} - \frac{1}{\theta_j}\right)} (\log_2 N)^{\sum_{j=2}^m \left(\frac{1}{p} - \frac{1}{\tau_j}\right)} << \sup_{f \in S_{p, \bar{\theta}, \bar{\tau}}^{\bar{r}} B} \|f - S_N^\gamma(f)\|_{\bar{p}}$$

**Proof.** Let  $f \in S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B$ . Now, since  $2 \leq p_j < +\infty$ ,  $2 \leq \tau_j < +\infty$ ,  $j = 1, \dots, m$ , using Theorem 1 and the inequality of different metric for trigonometric polynomials (see [5]), the inequality Holder's we obtain

$$\begin{aligned} \|f\|_{\bar{p}} &<< \left\{ \sum_{\bar{s} \in \mathbb{Z}^m} \|\delta_{\bar{s}}(f)\|_{\bar{p}}^2 \right\}^{\frac{1}{2}} << \\ &<< \left\{ \sum_{\bar{s} \in \mathbb{Z}^m} \|\delta_{\bar{s}}(f)\|_{\bar{p}, \bar{\theta}}^2 \prod_{j=1}^m (s_j + 1)^{\frac{1}{p_j} - \frac{1}{\theta_j}} \right\}^{\frac{1}{2}} << \\ &<< \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \left\| \left\{ 2^{-\langle \bar{s}, \bar{r} \rangle} \prod_{j=1}^m (s_j + 1)^{\frac{1}{p_j} - \frac{1}{\theta_j}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\epsilon}}}, \end{aligned} \quad (14)$$

where  $\bar{\epsilon} = (\epsilon_1, \dots, \epsilon_m)$ ,  $\epsilon_j = \frac{2\tau_j}{\tau_j - 2}$ ,  $j = 1, \dots, m$ . Taking into account that  $r_j > 0$ ,  $j = 1, \dots, m$  we get

$$\left\| \left\{ 2^{-\langle \bar{s}, \bar{r} \rangle} \prod_{j=1}^m (s_j + 1)^{\frac{1}{p_j} - \frac{1}{\theta_j}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\epsilon}}} < \infty.$$

Hence, it follows from (14) that  $S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B \subset L_{\bar{p}}(\mathbb{I}^m)$  and

$$\|f - S_N^{\gamma}(f)\|_{\bar{p}} << \left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f)\|_{\bar{p}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} \left\| \left\{ 2^{-\langle \bar{s}, \bar{r} \rangle} \prod_{j=1}^m (s_j + 1)^{\frac{1}{p_j} - \frac{1}{\theta_j}} \right\}_{\bar{s} \in \Gamma^{\perp}(N)} \right\|_{l_{\bar{\epsilon}}} \quad (15)$$

where  $\Gamma^{\perp}(N) = \{\bar{s} \in \mathbb{Z}_+^m : \langle \bar{s}, \bar{\gamma} \rangle \geq \log_2 N^{\frac{1}{r_1}}\}$ .

Next applying inequality

$$I_N = \left\| \left\{ 2^{-\langle \bar{s}, \bar{\gamma} \rangle \beta} \prod_{j=1}^m s_j^{d_j} \right\}_{\bar{s} \in \Gamma^{\perp}(N)} \right\|_{l_{\bar{\theta}}} << 2^{-n\beta} n^{\sum_{j=1}^m d_j + \sum_{j=2}^m \frac{1}{\theta_j}}$$

for  $\beta > 0$ ,  $d_j \geq 0$ ,  $j = 1, \dots, m$ , then

$$\begin{aligned} &\left\| \left\{ 2^{-\langle \bar{s}, \bar{\gamma} \rangle r_1} \prod_{j=1}^m (s_j + 1)^{\frac{1}{p_j} - \frac{1}{\theta_j}} \right\}_{\bar{s} \in \Gamma^{\perp}(N)} \right\|_{l_{\bar{\epsilon}}} << \\ &<< N^{-r_1} (\log_2 N)^{\sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{\theta_j}) + \sum_{j=2}^m \frac{1}{\epsilon_j}}. \end{aligned}$$

Therefore from (15) we obtain

$$\|f - S_N^{\gamma}(f)\|_{\bar{p}} << N^{-r_1} (\log_2 N)^{\sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{\theta_j}) + \sum_{j=2}^m (\frac{1}{2} - \frac{1}{\tau_j})}$$

for any function  $f \in S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B$ . This proves the upper bound.

Let us prove the lower bound. Consider the function

$$f_4(\bar{x}) = (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} \sum_{\langle \bar{s}, \bar{\gamma} \rangle = \log_2 N^{\frac{1}{r_1}}} \prod_{j=1}^m 2^{-s_j r_j} s_j^{-\frac{1}{\theta_j}} \sum_{\bar{k} \in \rho(\bar{s})} \prod_{j=1}^m (k_j - 2^{s_j - 1} + 1)^{\frac{1}{p} - 1} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

Then  $f_4 \in L_{\bar{p}, \bar{\theta}}(\mathbb{I}^m)$ . Now, by relation

$$\left\| \sum_{\bar{k} \in \rho(\bar{s})} \prod_{j=1}^m (k_j - 2^{s_j-1} + 1)^{\frac{1}{p}-1} e^{i\langle \bar{k}, \bar{x} \rangle} \right\|_{\bar{p}, \bar{\theta}} \asymp \prod_{j=1}^m (s_j + 1)^{\frac{1}{\theta_j}} \quad (16)$$

for  $1 < p_j < \infty$ ,  $1 < \theta_j < \infty$ ,  $j = 1, \dots, m$  and by Lemma 1 [4] we get

$$\left\| \left\{ 2^{\langle \bar{s}, \bar{r} \rangle} \|\delta_{\bar{s}}(f_4)\|_{\bar{p}, \bar{\theta}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}} << (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left\| \{1\}_{\bar{s} \in \mathcal{K}(N)} \right\|_{l_{\bar{\tau}}} \leq C,$$

where  $\mathcal{K}(N) = \{\bar{s} \in \mathbb{Z}_+^m : \langle \bar{s}, \bar{\gamma} \rangle = \log_2 N^{\frac{1}{r_1}}\}$ .

Hence the function  $C_4 f_4 \in S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B$ .

Since  $2 \leq p = p_1 = \dots = p_m < \infty$ , then by Littlewood-Paley theorem [21] we obtain

$$\|f_4 - S_N^\gamma(f_4)\|_p = \|f_4\|_p >> \left\| \left\{ \sum_{\bar{s} \in \mathcal{K}(N)} |\delta_{\bar{s}}(f_4)|^2 \right\}^{\frac{1}{2}} \right\|_p >> \left\{ \sum_{\bar{s} \in \mathcal{K}(N)} \|\delta_{\bar{s}}(f_2)\|_p^p \right\}^{\frac{1}{p}}.$$

By relation (16) for  $\theta_j = p_j = p$ ,  $j = 1, \dots, m$ , it follows that

$$\begin{aligned} \|f_4 - S_N^\gamma(f_4)\|_p &>> (\log_2 N)^{-\sum_{j=2}^m \frac{1}{\tau_j}} \left( \sum_{\bar{s} \in \mathcal{K}(N)} 2^{-\langle \bar{s}, \bar{r} \rangle p} \prod_{j=1}^m (s_j + 1)^{(\frac{1}{p} - \frac{1}{\theta_j})p} \right)^{\frac{1}{p}} >> \\ &>> N^{-r_1} (\log_2 N)^{\sum_{j=1}^m (\frac{1}{p} - \frac{1}{\theta_j})} (\log_2 N)^{\sum_{j=2}^m (\frac{1}{p} - \frac{1}{\tau_j})} \end{aligned}$$

for function  $C_4 f_4 \in S_{\bar{p}, \bar{\theta}, \bar{\tau}}^{\bar{r}} B$ . So Theorem 6 has been proved.

**Remark.** Note that for the case  $q_j = \theta_j = q$ ,  $p_j = p$ ,  $\tau_j = \tau$ ,  $j = 1, \dots, m$ , Theorem 5 was proved by S.A. Stasyuk [30]. For the case  $p_j = \theta_j^{(1)} = p$ ,  $q_j = \theta_j^{(2)} = q$ ,  $\tau_j = +\infty$ ,  $j = 1, \dots, m$ , Theorem 4 was proved by N.N. Pustovoirov [23].

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## REFERENCES

- [1] Akishev G. *Approximation of function classes in Lorentz spaces with mixed norm*, East Journal of Approx. - 2008 - Vol.14(2) - P. 193–214.
- [2] Akishev G. *Approximation of function classes in spaces with mixed norms*, Mathem. sb. 2006 - Vol. 197(8) - P. 17–40.
- [3] Akishev G. *On degree of approximation of function classes in the Lebesgue space with the anisotropic norm*, Uchenie Zapiskii Kazan Univer. - 2006. - Vol. 148(2). - P. 5–17.
- [4] Akishev G. *On order of approximation of generalized Nikol'skii-Besov class in Lorentz space*, Centre de Recerca Matematica, Barcelona - 2016 - Preprint 1222 - 22 p.
- [5] Akishev G., *Inequalities of distinct metric of polynomials in Lorentz spaces with mixed norm*, First Erjanov reading, Pavlodar state universuty, - 2004. - P.211–215.
- [6] Amanov T.I. *Representation and embedding theorems for the functional spaces  $S_{p,\theta}^r B(R^n)$  and  $S_{p,\theta}^r * B$* , Trudy Math. Inst. Steklov. - 1965. - Vol. 77. - P. 143–167.
- [7] Babenko K.I. *Approximation by trigonometric polynomials in a certain class of periodic functions of several variables*, Dokl. Akad. Nauk SSSR. - 1960. - Vol. 132(5). - P. 982–985. (English transl. in Soviet math. Dokl. 1960, V.1, 672–675).
- [8] Bary N.K., Stechkin S.B. (*), The best approximations and differential properties of two conjugate functions*, Trudy Moskov Mat. Sb.- 1956. - Vol.5. - P. 483–522.

- [9] Bazarkhanov D.B. *Approximation with wavelets and Fourier widths of classes of periodic functions of several variables I*, Trudy Math. Inst. Steklov. - 2010. - Vol. 269. - P. 8–30.
- [10] Bekmaganbetov K.A. *On orders of approximations of the Besov class in the metric of the anisotropic Lorentz space*, Ufinskii Matem. Journal. - 2009. - Vol.1(2). - P. 9–16.
- [11] Bekmaganbetov K.A. *Orders of approximations of Besov classes in the metric of anisotropic Lorentz spaces*, Methods of Fourier analysis and approximation theory, (Editors: Michael Ruzhansky and Sergey Tikhonov) , Birkhauser. - 2016 - P. 149–158.
- [12] Blozinski A.P. *Multivariate rearrangements and Banach function spaces with mixed norms*, Trans. Amer. Math. Soc. -1981. - Vol.263(1). - P. 146-167.
- [13] Bugrov Ya. S., *Approximation of function classes with the dominant mixed derivative*, Math. sb. - 1964. - Vol.64(3) - P. 410-418.
- [14] DeVore R.A., Konyagin S.V., Temlyakov V.N. *Hyperbolic wavelet approximation*, Construc. approx. - 1998. - Vol. 14. - P. 1-26.
- [15] Dinh Dung *Approximation by trigonometric polynomials of functions of several variables on the torus*, Math. sb. - 1986. - Vol. 131(2). - P. 251-271.
- [16] Galeev E.M. *Approximation of some classes of periodic functions of several variables by Fourier sums in the metric of  $\tilde{L}_p$* , Uspekhi Matem. Nauk. - 1977. - Vol.32(4). - P. 251– 252.
- [17] Galeev E.M. *Approximation by of Fourier sums of classes of functions with bounded derivatives*, Math. zametki. - 1978. - Vol.23(2). - P. 197-212.
- [18] Lizorkin P.I., Nikol'ski S.M. *Spaces of functions of mixed smoothness from the decomposition point of view*, Proc. Stekov Inst. Math.- 1989. - Vol. 187. - P. 143-161.
- [19] Mityagin B.S. *Approximation of functions in the spaces  $L_p$  and  $C$  on the torus*, Math. sb. - 1962. - Vol. 58(4). - P. 397–414.
- [20] Nikol'ski S. M. *Functions with the dominant mixed derivative which satisfy multi Hol'der's condition*, Sibirski Math. zhurnal.- 1963. - Vol.4(6). - P. 1342–1364.
- [21] Nikol'ski S. M. *Approximation of functions of several variables and embedding theorems*, Nauka, Moscow. - 1977. - 456 p.
- [22] Nikol'skaya N.S. (), *The approximation differentiable functions of several variables by Fourier sums in the  $L_p$ -metric*, Sibirski Math. zhurnal. - 1974. - Vol. 15(2). - P. 395–412.
- [23] Pustovoitov N. N. *Approximation of multidimensional functions with a given majorant of mixed moduli of continuity*, Matem. zametki. - 1999. - Vol. 65(1). - P. 107–117.
- [24] Pustovoitov N. N. *On best approximations by analogs of “proper” and “improper” hyperbolic crosses*, Math. Notes. -2013. - Vol. 93, 3. - P. 487-496.
- [25] Romanyuk A.S. *Approximation of the Besov classes of periodic functions of several variables in the space  $L_q$* , Ukrain . Math. J. - 1991. - Vol. 43(1). - P. 1297-1306.
- [26] Romanyuk A.S. *On estimates of approximation characteristics of the Besov classes of periodic functions of many variables*, Ukrain . Math. J. - 1997. - Vol. 9(9). - P. 1409-1422.
- [27] Sickel W., Ullrich T. *Tensor products of Sobolev – Besov spaces and applications to approximation from the hyperbolic cross*. Journal Approx. Theory. - 2009. - Vol. 161(2). - P. 748– 786.
- [28] Sikhov M.B. *Approximation of functions of several variables with a given majorant in the Besov space*. Math. Journal. - 2002. - Vol. 2(4). - P. 95–100.
- [29] Schmeisser H.-J., Sickel W. *Spaces of functions of mixed smoothness and approximation from hyperbolic crosses*, Journal of Approximation Theory. - 2004. - Vol. 128(2). - P. 115–150.
- [30] Stasyuk S. A. *Thr best approximations of periodic functions of several variables in the classes  $B_{p,\theta}^\Omega$* , Matem. zametki. - 2010. - Vol. 87(1). - P. 108 – 121.
- [31] Sun Yongsheng, Wang Heping. *Representation and approximation of multivariate periodic functions with bounded mixed moduli of smoothness*, Trudy Math. Inst. Steklov. - 1997. - Vol.219. - P. 356–377.
- [32] Telyakovskii S.A. *Some estimates for trigonometric series with quasiconvex coefficients*. Math. sb.- 1964. - Vol. 63(3). - P. 426–444.
- [33] Temlyakov V.N., *Approximation of functions with bounded mixed derivative*, Trudy Math. Inst. Steklov. - 1986. - Vol. 178. - P. 3–112 .
- [34] Temlyakov V.N. (), *Approximation of periodic functions of several variables with bounded mixed differences*, Mat. Sb. - 1980. - Vol. 113(1). - P. 65-80.
- [35] Tikhomirov V.M. *Approximation theory*. Itogy Nauki i Tekhniki : Sovrem. Probl. Math.: Fud. Naprav. VINITI, Moscow. - 1987. - Vol. 14. - P. 103–270.

- [36] Dinh Dung , Vladimir N. Temlyakov, Tino Ullrich. *Hyperbolic cross approximation*. arXiv: 1601.03978v1[math.NA] 15 Jan, 2016, 154 p.

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